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Simple Axioms for Orthomodular Implication Algebras

Ivan Chajda,[1] Radomír Halaš,[1] and Helmut Länger[2,3]

Simple, independent axioms for orthomodular implication algebras are presented.

KEY WORDS: implication algebra; orthoimplication algebra; orthomodular implication algebra; independent axioms; ortholattice.

1. INTRODUCTION

Abbott (1967) introduced implication algebras as groupoids $(A, \cdot)$ satisfying

(I1) $(xy)x = x$,
(I2) $(xy)y = (yx)x$, and
(I3) $x(yz) = y(xz)$.

These axioms reflect some important properties of implication in Boolean algebras. He further showed that there is a natural bijective correspondence between these groupoids and join semilattices with one every principal filter of which is a Boolean algebra. Abbott (1976) and Chajda, Halaš, and Länger (2001) generalized these ideas and results from Boolean algebras to orthomodular lattices. Abbott (1976) defined orthoimplication algebras as groupoids $(A, \cdot)$ satisfying

(OI1) $(xy)x = x$,
(OI2) $(xy)y = (yx)x$, and
(OI3) $x((yx)z) = xz$.

Whereas Abbott (1976) assumed a natural compatibility condition between the complements in different principal filters to hold this was not done by Chajda, Halaš, and Länger (2001).

[1] Department of Algebra and Geometry, Palacký University Olomouc, Olomouc, Czech Republic.
[2] Institut für Diskrete Mathematik Und Geometrie, Technische Universität Wien, Wien, Austria.
[3] To whom correspondence should be addressed; e-mail: h.laenger@tuwien.ac.at.

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2. THE ORIGINAL AXIOM SYSTEM

Chajda, Halaš, and Länger (2001) defined orthomodular implication algebras as algebras (A, · ,1) of type (2, 0) satisfying

(O1) $xx = 1$,
(O2) $x(yx) = 1$,
(O3) $(xy)x = x$,
(O4) $(xy)y = (yx)x$,
(O5) $(((xy)y)z)(xz) = 1$, and
(O6) $((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$.

These axioms are not independent since (O2) follows from (O1), (O3), and (O5):

$$x(yx) = ((xx)x)(yx) = (1x)(yx) = ((11)x)(yx) = ((((1y)1)1)x)(yx)$$
$$= (((((yy)y)1)1)x)(yx) = (((y1)1)x)(yx) = 1$$

That $xx = yy$ follows from (O3) and (O4) can be seen as follows:
We have

$$x(xy) = ((xy)x)(xy) = xy$$

and hence

$$xx = ((xy)x)x = (x(xy))(xy) = (xy)(xy)$$

from which we conclude

$$xx = (xy)(xy) = ((xy)y)((xy)y) = ((yx)x)((yx)x) = (yx)(yx) = yy$$

Since xx is an equational constant, (O1)–(O6) can be equivalently reformulated as axioms for groupoids. In this way, orthomodular implication algebras can be considered as groupoids satisfying

(O1′) $x(yx) = xx$,
(O2′) $(xy)x = x$,
(O3′) $(xy)y = (yx)x$,
(O4′) $(((xy)y)z)(xz) = xx$, and
(O5′) $((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$.

As above, $xx = yy$ follows from (O2′) and (O3′). Also the axioms (O1′)–(O5′) turn out not to be independent since (O1′) follows from (O2′)–(O4′):

$$x(yx) = ((xx)x)(yx) = (((yy)(yy))x)(yx) = (((((yy)y)(yy))(yy))x)(yx)$$
$$= (((y(yy))(yy))x)(yx) = yy = xx$$

3. THE NEW AXIOM SYSTEM

Now we state four simple axioms characterizing orthomodular implication algebras:

Theorem 3.1. *The axiom system* (O1′)–(O5′) *is equivalent to the following system of axioms*:

(O1″) $(xy)x = x$,
(O2″) $(xy)y = (yx)x$,
(O3″) $(((xy)y)z)(xz) = xx$, *and*
(O4″) $((((((xy)y)z)x)x)z)x)x = (((xy)y)z)z$.

Proof: If (O1′)–(O5′) hold then $xx = yy$ follows as above and one obtains

$$((xy)y)y = (y(xy))(xy) = (yy)(xy) = ((xy)(xy))(xy) = xy$$

and hence

$$((((((xy)y)z)x)x)z)x)x = ((((((((xy)y)z)z)z)x)x)z)x)x = (((xy)y)z)z$$

If, conversely, (O1″)–(O4″) hold then $xx = yy$, (O1′) and $((xy)y)y = xy$ follows as above and

$$((((((((xy)y)z)z)z)x)x)z)x)x = ((((((xy)y)z)x)x)z)x)x = (((xy)y)z)z \qquad \square$$

4. INDEPENDENCE OF THE NEW AXIOMS

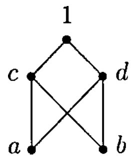
We now show that in contrast to the axioms (O1)–(O6) and (O1′)–(O5′), respectively, the axioms (O1″)–(O4″) are independent:

Theorem 4.2. *The axioms* (O1″)–(O4″) *are independent.*

Proof: The groupoid $(\{1, 2\}, \cdot)$ with $xy = 1$ for $x, y \in \{1, 2\}$ satisfies (O2″)–(O4″) but not (O1″).

The groupoid $(\{1, 2\}, \cdot)$ with $xy = x$ for $x, y \in \{1, 2\}$ satisfies (O1″), (O3″), and (O4″) but not (O2″).

Let $(P, \leq)$ denote the poset with the Hasse diagram

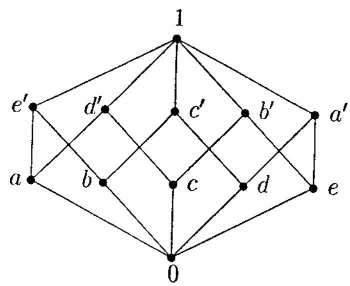


and for $x, y \in P$, let $x + y$ denote the supremum of x and y if it exists and 1 otherwise. Then the groupoid $(P, \cdot)$ where for $x, y \in P$, xy denotes the complement of $x + y$ in the Boolean algebra $[y, 1]$ satisfies (O1″), (O2″), and (O4″) but not (O3″) since

$$(((ac)c)b)(ab) = ((1c)b)b = (cb)b = db = c \neq 1 = aa$$

That $(P, \cdot)$ satisfies (O1″), (O2″), and (O4″) can be verified by using the fact that all principal filters of $(P, \leq)$ are Boolean algebras. This means that the only nontrivial cases for the axioms to verify are those where both a and b occur.

Finally, let $(L, \vee, \wedge^i, ', 1)$ denote the ortholattice with the Hasse diagram



Then the groupoid $(L, \cdot)$ where for $x, y \in L$, xy denotes the complement of $x \vee y$ in the ortholattice $[y, 1]$ satisfies (O1″)–(O3″) but not (O4″) since

$$\begin{aligned}((((((ae')e')0)a)a)0)a)a &= (((((1e')0)a)a)0)a)a = ((((e'0)a)a)0)a)a = \\ &= ((((ea)a)0)a)a = (((aa)0)a)a = ((10)a)a \\ &= (0a)a = 1a = a \neq e' = e0 = (e'0)0 \\ &= ((1e')0)0 = (((ae')e')0)0\end{aligned}$$

That $(L, \cdot)$ satisfies (O1″)–(O3″) can be verified by using the fact that all principal filters of $(L, \leq)$ are ortholattices. □

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